Lecture 2

↳ Multiple Systems

We can encode information in large single systems

↳ But its often more convenient to consider collections of small systems.

Again we start from classical systems

① Classical information

classical state sets

Suppose we have X with Σ & Y with Γ.

↳ we can consider the single system (X, Y)

what are its classical states?

→

<u>Answer</u>: Classical state set of (X, Y) is the <u>Cartesian product</u> of Σ & Γ

i.e., $\Sigma \times \Gamma = \{(a,b) : a \in \Sigma \text{ \& } b \in \Gamma\}$

//

If we say (X,Y) is in $(a,b) \in \Sigma \times \Gamma$ we mean X is in $\underset{\Sigma}{\underset{\Uparrow}{a}}$ & Y is in $\underset{\Gamma}{\underset{\Uparrow}{b}}$.

//

For more than two systems we can generalize straightforwardly...

Given $X_1, \ldots, X_n$ with $\Sigma_1, \ldots, \Sigma_n$ the classical state set of $(X_1, \ldots, X_n)$ is

$$\Sigma_1 \times \ldots \times \Sigma_n = \{(a_1, \ldots, a_n) \mid a_1 \in \Sigma_1 \ldots a_n \in \Sigma_n\}$$

String representation of states

↳ it is often convenient to denote the joint state $(a_1, \ldots, a_n)$ via $a_1 a_2 \ldots a_n$.

Example: bits (of course)

↳ consider $X_1, \ldots, X_{10}$ with

$$\Sigma_1 = \ldots = \Sigma_{10} = \{0,1\}$$

↳ states of the joint system $\in \Sigma_1 \times \times \Sigma_{10} = \{0,1\}^{10}$

length 10 strings using the binary alphabet!

↳
$$\left.\begin{array}{c} 0000000000 \\ 0000000001 \\ 0000000010 \\ \vdots \\ 1111111111 \end{array}\right\} 2^{10} \text{ states}$$

In general if we have an alphabet $\{a_1, \ldots, a_n\}$ then $\{a_1, \ldots a_n\}^m$ has n^m states.

↳ make sure you understand this!

Probabilistic States

↳ A probabilistic state of a joint system assigns a probability to each possible joint state

Example: X & Y are bits

$$\Sigma = \Gamma = \{0,1\}$$

$$\Pr((X,Y) = 00) = 1/2$$
$$\Pr((X,Y) = 01) = 0$$
$$\Pr((X,Y) = 10) = 0$$
$$\Pr((X,Y) = 11) = 1/2$$

Note X & Y are "correlated"
↳ they always agree

Ordering Cartesian products

↳ Just as in single systems we represent probabilistic states as vectors with an entry for each element of the state set.

↳ But we have to impose an order!

For cartesian products there is a convention!

↳
$$\Sigma_1 \times \Sigma_2 = \underbrace{\begin{array}{cc|c} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 2 & 0 & 3 \\ 2 & 1 & 4 \\ 3 & 0 & 5 \\ 3 & 1 & 6 \end{array}}$$

$$\Sigma_1 \times \Sigma_2 = \{1,2,3\} \times \{0,1\}$$

this is called "alphabetical ordering"

So, lets consider the system (X_1, X_2) with $\Sigma_1 = \Sigma_2 = \{0,1\}$

$\hookrightarrow$ state of the system is a probability vector $v \in \mathbb{R}^{2^2}$

$\hookrightarrow$

$$v = \begin{pmatrix} 1/2 \\ 0 \\ 0 \\ 1/2 \end{pmatrix} \begin{array}{l} \longrightarrow 00 \\ \longrightarrow 01 \\ \longrightarrow 10 \\ \longrightarrow 11 \end{array}$$

NB $\rightarrow$ we will work with bitstrings a lot.

$\hookrightarrow$ check you know $\{0,1\}^m$!

$\hookrightarrow$

Independence of two systems

↳ We say that the systems within a state of two systems are independent if learning something about the one yields no information about ~~the~~ other.

↳ lets be precise...

↳ Given X & Y with Σ & Γ we say X & Y are <u>independent</u> if

$$\Pr((X,Y)=ab) = [\Pr(X=a)] \times [\Pr(Y=b)]$$

for all $ab \in \Sigma \times \Gamma$.

↳

We can rephrase this in terms of probability vectors...

↳ Assume (X, Y) is in the state $|v\rangle$ with

$$|v\rangle = \sum_{a,b \in \Sigma \times \Gamma} p_{ab} |ab\rangle$$

X & Y are independent if $\exists$

$$\underbrace{|\phi\rangle = \sum_{a \in \Sigma} q_a |a\rangle}_{\text{state of } X} \quad \& \quad \underbrace{|\psi\rangle = \sum_{b \in \Gamma} r_b |b\rangle}_{\text{state of } Y}$$

such that $p_{ab} = q_a r_b$

↳

Examples

- (X, Y) with $\Sigma = \Gamma = \{0,1\}$

  ↳ $|v\rangle = \frac{1}{6}|00\rangle + \frac{1}{12}|01\rangle + \frac{1}{2}|10\rangle + \frac{1}{4}|11\rangle$

  ↳ X & Y are independent because

$$|\phi\rangle = \tfrac{1}{4}|0\rangle + \tfrac{3}{4}|1\rangle$$

$$|\psi\rangle = \tfrac{2}{3}|0\rangle + \tfrac{1}{3}|1\rangle$$

		q	r	p
0	0	1/4	2/3	1/6
0	1	1/4	1/3	1/12
1	0			⋮
1	1			

satisfies the condition!

- But for $|v\rangle = \frac{1}{2}|00\rangle + \frac{1}{2}|11\rangle$ the systems are <u>not</u> independent!

$\Rightarrow$ Suppose they were independent

↳ either $q_0 = 0$ or $r_1 = 0$ (otherwise $p_{01} \neq 0$)

↳ but then either p_{00} or $p_{11} = 0$

↳ but this is not the case!

We have defined <u>independent</u> systems

↳ Now, a <u>correlated system</u> is any system which is not independent.

<u>Tensor products of vectors</u>

↳ This notion will help us significantly.

<u>Defn</u>: Given $|\phi\rangle = \sum_{a \in \Sigma} \alpha_a |a\rangle$ & $|\psi\rangle = \sum_{b \in \Gamma} \beta_b |b\rangle$

the tensor product is

$$|\phi\rangle \otimes |\psi\rangle = \sum_{a,b \in \Sigma \times \Gamma} \alpha_a \beta_b |ab\rangle .$$

Equivalently, $|\pi\rangle = |\phi\rangle \otimes |\psi\rangle$ is defined via

$$\langle ab | \pi \rangle = \langle a | \phi \rangle \langle b | \psi \rangle$$

$\forall \; a \in \Sigma, \; b \in \Gamma.$

We now have an easy definition of independence...

↳ (X, Y) is an independent system if the joint vector $|\pi\rangle$ is a tensor product of prob vectors for X & Y,

↳ ie if $\underbrace{|\pi\rangle}_{\text{we call } |\pi\rangle \text{ a "product state"}} = |\phi\rangle \otimes |\psi\rangle$

we call $|\pi\rangle$ a "product state"

Notation

↳ we $\underbrace{\text{often}}_{\text{essentially always}}$ use $|\phi\rangle|\psi\rangle$ for $|\phi\rangle \otimes |\psi\rangle$

↳ essentially always

We also use alphabetical ordering for tensor products of column vectors

$$\begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_m \end{pmatrix} \otimes \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_k \end{pmatrix} = \begin{pmatrix} \alpha_1 \beta_1 \\ \vdots \\ \alpha_1 \beta_k \\ \alpha_2 \beta_1 \\ \vdots \\ \alpha_2 \beta_k \\ \vdots \\ \alpha_m \beta_1 \\ \vdots \\ \alpha_m \beta_k \end{pmatrix}$$

& we note that

$$\underbrace{|a\rangle} \otimes \underbrace{|b\rangle} = \underbrace{|ab\rangle}$$

standard basis vectors!

we sometimes write $|a\rangle \otimes |b\rangle = |a, b\rangle$

→

Properties of the tensor product

The tensor product is bilinear

① linearity in first argument

$$\left(|\phi_1\rangle + |\phi_2\rangle\right) \otimes |\psi\rangle = |\phi_1\rangle|\psi\rangle + |\phi_2\rangle|\psi\rangle$$

$$\left(\alpha|\phi\rangle\right) \otimes |\psi\rangle = \alpha|\phi\rangle|\psi\rangle$$

② linearity in second argument

$$|\phi\rangle \otimes \left(|\psi_1\rangle + |\psi_2\rangle\right) = |\phi\rangle|\psi_1\rangle + |\phi\rangle|\psi_2\rangle$$

$$|\phi\rangle \otimes \left(\alpha|\psi\rangle\right) = \alpha|\phi\rangle|\psi\rangle$$

NB $\left(\alpha|\phi\rangle\right) \otimes |\psi\rangle = |\phi\rangle \otimes (\alpha|\psi\rangle) = \alpha|\phi\rangle|\psi\rangle$

Independence & tensor products for more than two systems

We can generalize straightforwardly

Given $X_1 \dots X_n$ & $\Sigma_1 \dots \Sigma_n$ the joint state $|\psi\rangle$ is a product state if

$$|\psi\rangle = |\phi_1\rangle \otimes \quad \otimes |\phi_n\rangle$$

defined via

$$\langle a_1 \dots a_n|\psi\rangle = \langle a_1|\phi_1\rangle \dots \langle a_n|\phi_n\rangle$$

or

recursively via

$$|\phi_1\rangle \otimes \quad \otimes |\phi_n\rangle = \left(|\phi_1\rangle \times \quad \times |\phi_{n-1}\rangle\right) \otimes |\phi_n\rangle$$

now its <u>multilinear</u>

& $|a_1\rangle \otimes \quad \otimes |a_n\rangle = |a_1 \dots a_n\rangle = |a_1, \dots, a_n\rangle$

Measurements of probabilistic states

→ if we view a joint system as a single system then measurement is easy → we already know how.

→ **Example** (X, Y) in state

$$|\psi\rangle = \tfrac{1}{2}|00\rangle + \tfrac{1}{2}|11\rangle$$

outcome	Probability
00	1/2
01	0
10	0
11	1/2

→ once we see an outcome we are in the corresponding basis state.

Partial measurements

↳ There is something new here!

↳ we can choose to measure only part of a joint system!

↳ we will get an outcome for each measured system & our knowledge of remaining systems will be effected!

Simple case: Two systems & we measure only one

↳ X & Y with $\Sigma \times \Gamma$

↳ measure X

↳

Of course the probability to see $a \in \Sigma$ must be the same as if we had measured both systems ⑰

$$\Pr(X = a) = \underbrace{\sum_{b \in \Gamma} \Pr((X,Y) = (a,b))}_{\text{marginal formula for } X}$$

(marginalize over Y)

Makes intuitive sense

↳ Probabilities reflect our belief about some definite state of the system.

Now, we have to update our description of Y given what we learned about X

$$\Pr(Y=b \mid X=a) = \frac{\Pr((X,Y)=(a,b))}{\Pr(X=a)}$$

$\underbrace{\Pr(Y=b \mid X=a)}_{\text{conditional probability}}$

In terms of probability vectors...

$$|\psi\rangle = \sum_{a,b \in \Sigma \times \Gamma} p_{ab} |ab\rangle$$

$$\Pr(X=a) = \sum_{b \in \Gamma} p_{ab}$$

$\therefore$ reduced state of X is

$$|\psi\rangle = \sum_{a \in \Sigma} \left(\sum_{c \in \Gamma} p_{ac} \right) |a\rangle$$

if we obtain outcome $a \in \Sigma$ when we measure X, then probabilistic state of Y is updated via

$$|\pi_a\rangle = \frac{\sum_{b \in \Gamma} p_{ab} |b\rangle}{\left[\sum_{c \in \Gamma} p_{ac}\right] Pr(X = a)}$$

So, if we measure X & see $a \in \Sigma$ the updated joint state of the system is

$$|\psi\rangle = |a\rangle \otimes |\pi_a\rangle$$

We can think of this factor as a "normalization" constant which accounts for the knowledge we gained learning about X.

↳ useful for calculations

Example: X with $\Sigma = \{0,1\}$
Y with $\Gamma = \{1,2,3\}$

assume $|\psi\rangle = \frac{1}{2}|01\rangle + \frac{1}{12}|03\rangle + \frac{1}{12}|11\rangle + \frac{1}{6}|12\rangle + \frac{1}{6}|13\rangle$

$$= \begin{array}{c c} \begin{pmatrix} 1/2 \\ 0 \\ 1/12 \\ 1/12 \\ 1/6 \\ 1/6 \end{pmatrix} & \begin{array}{c} 0\ 1 \\ 0\ 2 \\ 0\ 3 \\ 1\ 1 \\ 1\ 2 \\ 1\ 3 \end{array} \end{array}$$

lets work through what happens if X is measured..

Note that we can use bilinearity to write

$$|\psi\rangle = |0\rangle \otimes \left(\frac{1}{2}|1\rangle + \frac{1}{12}|3\rangle\right) + |1\rangle \otimes \left(\frac{1}{12}|1\rangle + \frac{1}{6}|2\rangle + \frac{1}{6}|3\rangle\right)$$

! can always do this !

Now analysis is easy:

$$Pr(X=0) = \tfrac{1}{2} + \tfrac{1}{12} = \tfrac{7}{12}$$
$$Pr(X=1) = \tfrac{1}{12} + \tfrac{1}{6} + \tfrac{1}{6} = \tfrac{5}{12}$$

$\frac{7}{12} + \frac{5}{12} = 1$ good!

If we see $X = 0$

↳ state of $Y \Rightarrow$
$$\frac{\overset{p_{01}}{\tfrac{1}{2}|1\rangle} + \overset{p_{03}}{\tfrac{1}{12}|3\rangle}}{7/12} \quad (\Pr(X=0))$$

If we see $X = 1$

↳ state of $Y \Rightarrow$
$$\frac{\tfrac{1}{12}|1\rangle + \tfrac{1}{6}|2\rangle + \tfrac{1}{6}|3\rangle}{5/12}$$

Operations on probabilistic states of joint systems

↳ again, viewing as a single system makes life easier...

↳ operations are represented by stochastic matrices (with rows & columns indexed by $\Sigma \times \Gamma$)

↳ example

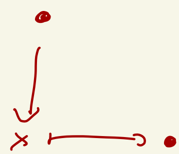
Example: X & Y are bits

2) if $X = 1$
  ↳ perform NOT on Y

else
  ↳ do nothing

"controlled not"
→ X is control bit
→ Y is target

$$\text{CNOT} \; (X \text{ control}, Y \text{ target}) = \begin{array}{c|c} & \begin{array}{cccc} 00 & 01 & 10 & 11 \end{array} \\ \hline \begin{array}{c} 00 \\ 01 \\ 10 \\ 11 \end{array} & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \end{array}$$



$|00\rangle \mapsto |00\rangle$
$|01\rangle \mapsto |01\rangle$
$|10\rangle \to |11\rangle$
$|11\rangle \mapsto |10\rangle$

$$= \underset{\text{row}}{|00\rangle}\underset{\text{column}}{\langle 00|} + |01\rangle\langle 01| + |11\rangle\langle 10| + |10\rangle\langle 11|$$

→ if Y is control & X is target

$$\text{CNOT} \; (X \text{ target}, Y \text{ control}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

↳ check you understand this!

Example

↳ Perform one of the following operations with probability $1/2$

- Set Y to be equal to X
- Set X to be equal to Y

$$\underbrace{\begin{pmatrix} 1 & 1/2 & 1/2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1/2 & 1/2 & 1 \end{pmatrix}}_{\text{write this in bra-ket notation}} = \frac{1}{2} \underbrace{\begin{array}{c} \begin{array}{cccc} 00 & 01 & 10 & 11 \end{array} \\ \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \end{array}} + \frac{1}{2} \underbrace{\begin{array}{c} \begin{array}{cccc} 00 & 01 & 10 & 11 \end{array} \\ \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \end{array}}$$

$|00\rangle \mapsto |00\rangle$
$|01\rangle \mapsto |00\rangle$
$|10\rangle \mapsto |11\rangle$
$|11\rangle \mapsto |11\rangle$

Set $Y = X$

$|00\rangle \mapsto |00\rangle$
$|10\rangle \mapsto |00\rangle$
$|01\rangle \mapsto |11\rangle$
$|11\rangle \mapsto |11\rangle$

Set $X = Y$

Independent operations

↳ what if we act on X with M

on Y with N

stochastic matrices representing the operation

↳ what is the stochastic matrix for the joint operation??

↳ we need tensor product for matrices!

↳

Matrix tensor product

$$M = \sum_{a,b\in\Sigma} \alpha_{ab} |a\rangle\langle b|$$

$$N = \sum_{c,d\in\Gamma} \beta_{cd} |c\rangle\langle d|$$

$$M \otimes N = \sum_{a,b\in\Sigma} \sum_{c,d\in\Gamma} \alpha_{ab}\beta_{cd} |ac\rangle\langle bd|$$

or equivalently

$$\langle \underset{\text{row}}{ac} | M\otimes N | \underset{\text{column}}{bd} \rangle = \langle a|M|b\rangle\langle c|N|d\rangle$$

for all $a,b\in\Sigma$ & $c,d\in\Gamma$.

or equivalently, th unique matrix satisfying

$$M\otimes N(|\phi\rangle\otimes|\psi\rangle) = (M|\phi\rangle)\otimes(N|\psi\rangle)$$

for all $|\phi\rangle$ & $|\psi\rangle$.

We also have this convention for matrices

$$\begin{pmatrix} \alpha_{11} & \cdots & \alpha_{1m} \\ & & \\ \alpha_{m1} & & \alpha_{mm} \end{pmatrix} \otimes \begin{pmatrix} B_{11} & \cdots & B_{1k} \\ & & \\ B_{k1} & \cdots & B_{kk} \end{pmatrix}$$

$$= \begin{pmatrix} \begin{pmatrix} \alpha_{11} B_{11} & \cdots & \alpha_{11} B_{1k} \\ & & \\ \alpha_{11} B_{k1} & \cdots & \alpha_{11} B_{kk} \end{pmatrix} & \cdots & \begin{pmatrix} \alpha_{1m} B_{11} & \cdots & \alpha_{1m} B_{1k} \\ & & \\ \alpha_{1m} B_{k1} & \cdots & \alpha_{1m} B_{kk} \end{pmatrix} \\ \vdots & & \vdots \\ \begin{pmatrix} \alpha_{m1} B_{11} & \cdots & \alpha_{m1} B_{1k} \\ & & \\ \alpha_{m1} B_{k1} & \cdots & \alpha_{m1} B_{kk} \end{pmatrix} & \cdots & \begin{pmatrix} \alpha_{mm} B_{11} & \cdots & \alpha_{mm} B_{1k} \\ \vdots & & \vdots \\ \alpha_{mm} B_{k1} & & \alpha_{mm} B_{kk} \end{pmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} (\alpha_{11} B) & \cdots & (\alpha_{1m} B) \\ \vdots & & \vdots \\ (\alpha_{m1} B) & \cdots & (\alpha_{mm} B) \end{pmatrix}$$

We define tensor products of multiple matrices analogously...

$$\mapsto \quad M_1 \otimes \quad \otimes M_m$$

$$\mapsto \quad \langle a_1 \ldots a_m | M_1 \otimes \quad \otimes M_m | b_1 \ldots b_m \rangle$$

$$= \langle a_1 | M_1 | b_1 \rangle \cdots \langle a_n | M_n | b_n \rangle$$

$$\forall \quad a_1, b_1 \in \Sigma_1 \ \ldots \ a_m, b_m \in \Sigma_m$$

We can also do it recursively!

Finally, tensor product is multiplicative

$$(M_1 \otimes \quad \otimes M_n)(N_1 \otimes \quad \otimes N_n)$$

$$= (M_1 N_1) \otimes \quad \otimes (M_n N_n)$$

$\forall\ M_i \otimes N_i$ (for which dimensions match)

Independent operations continued

if M acts on X & N acts on Y

↳ joint operation is $M \otimes N$ on (X, Y)

NB for states & operations tensor products represent independence!

→ if X & Y are independently in $|\phi\rangle$ & $|\psi\rangle$

↳ joint state $= |\phi\rangle \otimes |\psi\rangle$

→ if we act independently on X & Y

↳ joint operation is a tensor product $M \otimes N$

Example

↳ consider $\begin{pmatrix} 1 & 1/2 \\ 0 & 1/2 \end{pmatrix} = M$

↳ $|0\rangle \mapsto |0\rangle$

$|1\rangle \mapsto \frac{1}{2}|0\rangle + \frac{1}{2}|1\rangle$

if we apply M to X ⊗ NOT to Y

↳ $$\begin{pmatrix} 1 & 1/2 \\ 0 & 1/2 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 & 0 & 1/2 \\ 1 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1/2 \\ 0 & 0 & 1/2 & \end{pmatrix}}_{\text{stochastic!}}$$

Note, if we only act on X, that means acting on Y with the identity

$$\begin{pmatrix} 1 & 1/2 \\ 0 & 1/2 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \text{Do it} \end{pmatrix}$$